

Fluctuations of the vortex line density in turbulent flows of quantum fluids

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We present an analytical study of fluctuations of the Vortex Line Density (VLD) $\langle \delta\mathcal{L}(\omega) \delta\mathcal{L}(-\omega) \rangle$ in turbulent flows of quantum fluids. Two cases are considered. The first one is the counterflowing (Vinen) turbulence, where the vortex lines are disordered, and the evolution of quantity $\mathcal{L}(t)$ obeys the Vinen equation. The second case is the quasi-classic turbulence, where vortex lines are believed to form the so called vortex bundles, and their dynamics is described by the HVBK equations. The latter case, is of a special interest, since a number of recent experiments demonstrate the $\omega^{-5/3}$ dependence for spectrum VLD, instead of $\omega^{1/3}$ law, typical for spectrum of vorticity. In steady states the VLD is related to the normal velocity as $\mathcal{L} = (\rho\gamma/\rho_s)^2 v_n^2$ for the Vinen case, and $\mathcal{L} = |\nabla \times \mathbf{v}_n|/\kappa$ for rotating vortex tubes. In nonstationary situation, in particular, in the fluctuating turbulent flow there is a retardation between the instantaneous value of the normal velocity and the quantity \mathcal{L} . This retardation tends to decrease in the accordance with the inner dynamics, which has a relaxation character. In both cases the relaxation dynamics of VLD is related to fluctuations of the relative velocity, however if for the Vinen case the rate of temporal change for $\mathcal{L}(t)$ is directly depends on $\delta\mathbf{v}_{ns}$, for the HVBK dynamics it depends on $\nabla \times \delta\mathbf{v}_{ns}$. As a result, for the disordered case the spectrum $\langle \delta\mathcal{L}(\omega) \delta\mathcal{L}(-\omega) \rangle$ coincides with the spectrum $\omega^{-5/3}$. In the case of the bundle arrangement, the spectrum of the VLD varies (at different temperatures) from $\omega^{1/3}$ to $\omega^{-5/3}$ dependencies. This conclusion may serve as a basis for the experimental determination of what kind of the turbulence is implemented in different types of generation.

Keywords: superfluidity, vortices, quantum turbulence.

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I. INTRODUCTION

The problem of modeling classical turbulence with a set of chaotic quantized vortices is the hottest topic in modern studies of vortex tangles in quantum fluids (see e.g., recent reviews articles [1],[2],[3]). The most common view of quasi-classical turbulence is the model of vortex bundles. The point is that the quantized vortices have a fixed core radius, so they do not possess the important property of classical turbulence – stretching of tubes – which is responsible for the turbulence energy

cascade from the large scales to the small scales. The collections of the near-parallel quantized vortices (vortex bundles) do possess this property, so the idea that quasi-classical turbulence in quantum fluids is realized via vortex bundles of different sizes and intensities (number of threads) seems to be quite natural.

Recently two numerical evidences of the vortex bundles structures were obtained. Thus, in one numerical work, [4], in 2048^3 simulation of quantum turbulence within the Gross-Pitaevskii equation, authors observed nonuniform structures. The authors claimed that "the visualization of vortices clearly shows the bundle-like structure, which has never been confirmed in GPE simulations on smaller grids." In the other numerical work [5],[6], the authors studied the evolution of the vortex structures (at zero temperature) on the basis of the Biot-Savart law. They also observed structures reminiscent of the field of vorticity in classical turbulence (see e.g., [7]).

As for experimental confirmation, there are not so far strong evidences of the bundle structure. On the contrary, there are experimental results, which would seem to refute the idea of bundles. Thus, in experiments by Roche et al. [8], and by Bradley et al.[9], it was observed that the spectrum of the fluctuation of the VLD \mathcal{L} is compatible with a $-5/3$ power law. This contradicts the idea of the bundles structure, since the spectrum of the vorticity (and, correspondingly, of the VLD \mathcal{L} (via Eq. (8))) should scale as $1/3$ power law. An explanation was offered by Roche and Barenghi [10]. The authors considered the VLD \mathcal{L} to be decomposed into two components. The one, smaller, part consisting of the polarized component, is responsible for the large scale turbulent phenomena, whereas the other, disordered, part evolves as a "passive" scalar, thereby taking the $-5/3$ velocity spectrum. In paper [11], the authors performed the direct numerical simulations of the "truncated HVBK" model. They confirmed the existence of the $\omega^{-5/3}$ spectrum for fluctuation of the VLD, however, for the larger temperature this dependence became more shallow (probably reaching $\omega^{1/3}$), as it should be for the classic turbulence.

We present an analytical evaluation of the spectrum of fluctuations $\text{VLD} < \delta\mathcal{L}(\omega) \delta\mathcal{L}(-\omega) >$. Two cases are considered. The first one is the counterflowing (Vinen) turbulence, where vortex lines are disordered and dynamics of quantity $\mathcal{L}(t)$ is governed by the Vinen equation. The second case is the quasi-classic turbulence, where the vortex lines are believed to form the so called vortex bundles, and the dynamics of VLD obeys the HVBK equations. In steady states the VLD is related to the normal velocity as $\mathcal{L} = (\rho\gamma/\rho_s)^2 v_n^2$ for the Vinen case and $\mathcal{L} = |\nabla \times \mathbf{v}_n|/\kappa$ for the rotating vortex tubes (notations are standard, see e.g. [12]). In nonstationary situation, in particular, in the fluctuating turbulent flow there is a retardation between instantaneous value of the normal velocity and the quantity $\mathcal{L}(t)$. This retardation tends to decrease, according to the inner dynamics, which

has a relaxation character. In both cases, the relaxation of $\delta\mathcal{L}(t)$ is related to fluctuations of the normal velocity $\delta\mathbf{v}_{ns}$ velocity. If, however, for the Vinen disordered turbulence, the rate of temporal change for $\mathcal{L}(t)$ is directly depends on $\delta\mathbf{v}_{ns}$, for the HVBK dynamics it depends on the quantity $\nabla \times \delta\mathbf{v}_{ns}$. In addition, the relaxation mechanisms, and, consequently, times of relaxation are different. The factors, outlined above lead to different formulas for spectra $\langle \delta\mathcal{L}(\omega) \delta\mathcal{L}(-\omega) \rangle$ and their dependence on temperature.

II. VINEN EQUATION CASE

Let us study reaction of the vortex line density, in fluctuating flow of normal velocity supposing that the dynamics of $\mathcal{L}(t)$ obeys the Vinen equation

$$\frac{\partial\mathcal{L}}{\partial t} = \alpha_V |\mathbf{v}_{ns}| \mathcal{L}^{3/2} - \beta_V \mathcal{L}^2. \quad (1)$$

Equation (1) was derived phenomenologically initially for pure counterflowing superfluid helium [13–16]. Attempts to derive it an analytic form [17–20] demonstrated that this equation is seemingly valid for any non-structured turbulence. Under term "non-structured turbulence" we understand the vortex tangle, which consists of closed vortex loops of different sizes, uniformly distributed in space. It differs, for instance, from the turbulent fronts in rotating fluids, which deals with the lines terminating on lateral walls. It also differs from the mechanically excited turbulence, which is believed to consists of the so called vortex bundles composed of very polarized vortex filaments. Beside of the counterflow turbulence, the "non-structured turbulence" is generated by intensive sounds (both by the first and second). The case of vortex tangles. which appear also after the quench due to the Kibble-Zurek mechanism, or by the proliferation of vortices when approaching the critical temperature.

Our goal now is to study the stochastic properties of $\mathcal{L}(t)$, when \mathbf{v}_n fluctuates with the a given spectral density $\langle \delta\mathbf{v}_n(\omega) \delta\mathbf{v}_n(-\omega) \rangle = f(\omega)$. Further, for simplicity, we will study the pure counterflowing case in sense that the net flow is absent, $\mathbf{j} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s = 0$. Then, the average value \mathcal{L}_0 of the vortex line density is related to a relative velocity $\mathbf{v}_{ns} = \mathbf{v}_n - \mathbf{v}_n$ by the usual relation

$$\mathcal{L}_0 = \frac{\alpha_V^2}{\beta_V^2} \mathbf{v}_{ns}^2 = \gamma^2 \mathbf{v}_{ns}^2. \quad (2)$$

To take into account the fluctuations, we put

$$\mathcal{L} = \gamma^2 v_{ns}^2 + \delta\mathcal{L}, \quad \mathbf{v}_n = \mathbf{v}_{n0} + \delta\mathbf{v}_n. \quad (3)$$

From condition $\mathbf{j}=0$ it follows

$$\mathbf{v}_s = \mathbf{v}_{s0} - \frac{\rho_n}{\rho_s} \delta \mathbf{v}_n, \quad \mathbf{v}_{ns} = \mathbf{v}_{ns0} + \frac{\rho}{\rho_s} \delta \mathbf{v}_n. \quad (4)$$

Substituting Eqs. (3) and (4) into the Vinen equation (1) we arrive at

$$\frac{\partial \delta \mathcal{L}}{\partial t} = \alpha_V \frac{\rho}{\rho_s} \mathcal{L}_0^{3/2} \delta \mathbf{v}_n - \beta_V \frac{1}{2} \mathcal{L}_0 \delta \mathcal{L}. \quad (5)$$

Equation (5) shows, that the evolution of fluctuating part of the vortex line density $\delta \mathcal{L}$ bears the relaxation-type character with a characteristic time $\tau_V = 2/(\beta_V \mathcal{L}_0)$ and with a "force" proportional to $\delta \mathbf{v}_n$. It allows to express the spectrum of VLD $\langle \delta \mathcal{L}(\omega) \delta \mathcal{L}(-\omega) \rangle$ via the spectrum of a normal velocity. We accept further the usual in theory of turbulence relationship $k = \omega/v_n$ between the wavenumber k and the frequency ω (v_n is the mean flow). In the Fourier transforms Eq. (5) takes a form

$$i\omega \delta \mathcal{L}(\omega) = \alpha_V \frac{\rho}{\rho_s} \mathcal{L}_0^{3/2} \delta \mathbf{v}_n(\omega) - \beta_V \frac{1}{2} \mathcal{L}_0 \delta \mathcal{L}(\omega), \quad (6)$$

therefore the spectrum $\langle \delta \mathcal{L}(\omega) \delta \mathcal{L}(-\omega) \rangle$ is

$$\langle \delta \mathcal{L}(\omega) \delta \mathcal{L}(-\omega) \rangle = \frac{4(\alpha_V/\beta_V)^2 (\frac{\rho}{\rho_s})^2 \mathcal{L}_0 \langle \delta \mathbf{v}_n(\omega) \delta \mathbf{v}_n(-\omega) \rangle}{1 + (\omega \tau_V)^2}. \quad (7)$$

Relation (7) shows that for small frequencies, $\omega < 1/\tau$, the spectrum of the VLD reproduces the spectrum of fluctuations of the normal component, and if the Kolmogorov-type turbulence is developed in the normal component, then quantity $\langle \delta \mathcal{L}(\omega) \delta \mathcal{L}(-\omega) \rangle$ scales as $\omega^{-5/3}$. Interestingly, if we accept the conditions of the experiment by Roche et al.[8], then (e.g. for the mean flow $v_n \approx 1$ m/s at $T = 1.6$ K) we have

$$\langle \delta \mathcal{L}(\omega) \delta \mathcal{L}(-\omega) \rangle \approx 4 * 10^{22} \omega^{-5/3},$$

which is close to the experimental data in order of magnitude. Note also the dependence on the applied velocity (about $\propto v_n^4$) is also consistent with experimental data.

III. HVBK CASE

In the case of quasi-classical turbulence, the set of vortex line is believed to consist of the many bundles containing a large number of threads inside of them. The macroscopic behavior of these bundles is quite similar to the dynamics of eddies in ordinary fluids. The coarse-grained hydrodynamic of the vortex bundles is studied by many authors (see e.g., [21],[22],[23],[24]),

but the basis of these studies was the hydrodynamics of rotating superfluids, or the Hall-Vinen-Bekarevich-Khalatnikov (HVBK) model (see e.g., book [25]). In the vortex bundles, the vorticity field of ω_s (to be exact, its absolute value) and the vortex line density \mathcal{L} are related to each other with the use of the Feynman's rule:

$$\omega_s = \kappa \mathcal{L}. \quad (8)$$

This formula reflects the fact that the VLD \mathcal{L} , in this case coincides with the areal density of lines in a plane perpendicular to the bundle. In terms HVBK dynamics of the vorticity obeys to the following equation (see [25])

$$\frac{\partial \omega_s}{\partial t} = \nabla \times [\mathbf{v}_L \times \omega_s], \quad (9)$$

where \mathbf{v}_L is the velocity of lines,

$$\mathbf{v}_L = \mathbf{v}_s + \alpha [\hat{\omega}_s \times (\mathbf{v}_n - \mathbf{v}_s)]. \quad (10)$$

The meaning of Eq. (9) is that it describes the motion of vortex lines in the transverse (with respect to the unit vector $\hat{\omega}_s$ along the vorticity) direction, when the coarse-grained superfluid velocity \mathbf{v}_s differs from the normal velocity \mathbf{v}_n . This equation is derived again without fluctuations. To take into account the latter, we use the formula (3) into the equation (9) and have

$$\frac{\partial \hat{\omega}_s (\mathcal{L} + \delta \mathcal{L})}{\partial t} = \nabla \times [(\mathbf{v}_{ns} + \delta \mathbf{v}_{ns})(\mathcal{L} + \delta \mathcal{L})].$$

After little algebra this equations is transformed to

$$\frac{\partial \hat{\omega}_s (\delta \mathcal{L})}{\partial t} = \alpha \mathcal{L} \nabla \times \delta \mathbf{v}_{ns} \quad (11)$$

The HVBK theory describes the redistribution of preexisting vortex lines in the transverse direction, it does not include a mechanism of the appearance of new lines. In papers [26],[27],[28] it was shown that the bundle structure of quantized vortices develops inside the eddies of the normal component due to proliferation of vortex filaments. Mechanism of this proliferation is quite involved, it reminds the developing of the vortex tangle in an applied counterflow with the growth of the number lines due to reconnections, and with the growth of length due to relative velocity with the subsequent polarization. We take into account this process by adding a relaxation-type term in Eq. (11):

$$\frac{\partial \hat{\omega}_s(\delta \mathcal{L})}{\partial t} = \alpha \mathcal{L} \nabla \times \delta \mathbf{v}_{ns} + \frac{1}{\tau_s} \delta \mathcal{L}. \quad (12)$$

Here τ_s is the time of relaxation of the bundle structure as described by Samuels et al. [26],[27],[28].

In Fourier the component we have equation

$$-i\omega \delta \mathcal{L}(\omega) = i\alpha \mathcal{L} \frac{\rho}{\rho_s} (\mathbf{k} \times \delta \mathbf{v}_{ns}) + \frac{1}{\tau_s} \delta \mathcal{L},$$

which lead to the following spectral density:

$$\langle \delta \mathcal{L}(\omega) \delta \mathcal{L}(-\omega) \rangle = \frac{\alpha^2 \mathcal{L}^2 \left(\frac{\rho}{\rho_s}\right)^2 \tau_s^2 k^2 \langle \delta \mathbf{v}_n(\omega) \delta \mathbf{v}_n(-\omega) \rangle}{1 + (\omega \tau_s)^2} \quad (13)$$

Again, as in the case of Vinen turbulence, the shape of the spectrum depends on the value of $\omega \tau_s$. The question of the relaxation time is a bit vague and unclear. Analyzing the numerical results, Samuels [26] offered an expression for τ_s , which included the circulation of the vortex tube (of normal component) and its size. Therefore it is impossible to apply directly his result. The important fact is that quantity τ_s is proportional $1/\sqrt{\alpha}$, thus, it decreases with increasing temperature.

For large τ_s , which implies a small coupling between the normal and superfluid components and is realized for a small temperature, the spectrum $\langle \delta \mathcal{L}(\omega) \delta \mathcal{L}(-\omega) \rangle \propto \langle \delta \mathbf{v}_n(\omega) \delta \mathbf{v}_n(-\omega) \rangle$, ($\omega^{-5/3}$ for Kolmogorov turbulence in the normal component). In this case the result is similar to the previous (Vinen turbulence) case, considered above. Even the value of spectrum is close to the one, which is induced by the counterflowing turbulence. However, for small $\omega \tau_s \ll 1$, which corresponds to large temperature (strong coupling due to the large mutual friction) the spectrum behaves as $\langle \delta \mathcal{L}(\omega) \delta \mathcal{L}(-\omega) \rangle \propto \omega^2 \langle \delta \mathbf{v}_n(\omega) \delta \mathbf{v}_n(-\omega) \rangle$, ($\omega^{1/3}$ for Kolmogorov turbulence), and the intensity of this spectrum is much lower. This result is in good qualitative agreement with the numerical result [11],[6],[29]). As for the quantitative analysis, there is problem of determination of the relaxation time τ_s , which is beyond the scope of the present study.

IV. DISCUSSION AND CONCLUSION

We studied analytically the spectrum of fluctuations of the vortex line density both in the counterflowing (Vinen) turbulence and on the basis of the HVBK theory. In both case these deviations of quantity $\delta \mathcal{L}(t)$ appear due to strongly fluctuating field of the normal velocity. Deviations of $\delta \mathcal{L}(t)$

evolve in the relaxation like manner, which is determined either from the Vinen equation or from the HVBK equation. It allows to find Fourier transform and evaluate the spectra $\langle \delta \mathcal{L}(\omega) \delta \mathcal{L}(-\omega) \rangle$. The crucial difference between these two cases is that the stirring force for $\mathcal{L}(t)$ in the Vinen case is proportional to perturbations of the normal velocity $\delta \mathbf{v}_n(t)$, whereas in the HVBK this force is related to $\nabla \times \delta \mathbf{v}_n(t)$. This difference results in different spectra and their dependence on the temperature. From analysis of the final relations for spectral densities $\langle \delta \mathcal{L}(\omega) \delta \mathcal{L}(-\omega) \rangle$ we can conclude that for the small temperature, both model offer the $\omega^{-5/3}$ spectrum, whereas for large temperature the HVBK theory results in the $\omega^{1/3}$ dependence. This conclusion may serve as a basis for experimental determination of what kind of turbulence is realized in different types of generation.

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